Chapter 5. Gravitation

Notes:

- *Most of the material in this chapter is taken from Young and Freedman, Chap. 13.*

5.1 Newton's Law of Gravitation

We have already studied the effects of gravity through the consideration of the gravitational acceleration on earth g and the associated potential gravitational energy U_{grav}. We now broaden our study and consider gravitation in a more general manner through the **Law of Gravitation** enunciated by Newton in 1687

Every particle of matter in the universe attracts every other particle with a force that is directly proportional to the product of the masses of the particles and inversely proportional to the square of the distance between them.

Mathematically, this law, and the magnitude of the force due to the gravitational interaction between two particles, is expressed with

$$F_{\text{grav}} = \frac{Gm_1m_2}{r^2}, \tag{5.1}$$

where $G = 6.67 \times 10^{-11}\ \text{N} \cdot \text{m}^2/\text{kg}^2$ is the **universal gravitational constant**, m_1 and m_2 the particles masses, and r the distance between them. Equation (5.1) is however not complete since the force due to gravitation is vectorial in nature. If we define the vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ for the positions of m_1 and m_2, respectively, and the unit vector that points from $\mathbf{r}_1$ to $\mathbf{r}_2$ is $\mathbf{e}_r = (\mathbf{r}_2 - \mathbf{r}_1)/|\mathbf{r}_2 - \mathbf{r}_1|$, then the force that m_2 exerts on m_1 is

$$\begin{aligned} \mathbf{F}_{21} &= F_{\text{grav}}\mathbf{e}_r \\ &= \frac{Gm_1m_2}{r^2}\mathbf{e}_r. \end{aligned} \tag{5.2}$$

Conversely, accordingly to Newton's Third Law, the force that m_1 exerts on m_2 is

$$\begin{aligned} \mathbf{F}_{12} &= -\mathbf{F}_{21} \\ &= -\frac{Gm_1m_2}{r^2}\mathbf{e}_r. \end{aligned} \tag{5.3}$$

Equations (5.2) and (5.3) taken together show that gravitation is an attractive force, i.e., the two masses are drawn to one another.

5.1.1 Gravitational Interactions with Spherically Symmetric Bodies

Although the related calculations are beyond the scope of our studies, it can be shown that *gravitational interactions involving spherically symmetric bodies (e.g., uniform spheres, cavities, and shells) can be treated as if all the mass was concentrated at the centres of mass of the bodies* (see Figure 1). This property is especially useful when dealing with the earth, other planets and satellites, as well as the sun and stars, which can all be approximated as being spherically symmetric.

There are other interesting results that can be derived for spherically symmetric bodies. For example, the gravitational force felt by a mass m located at a radius r inside a spherical shell of radius R and mass M (i.e., with $r \leq R$) is zero. That is,

$$\mathbf{F}_{\text{grav}} = \begin{cases} 0, & r < R \\ -\dfrac{GmM}{r^2}\mathbf{e}_r, & r \geq R \end{cases} \tag{5.4}$$

with $\mathbf{e}_r$ the unit vector directed from the centre of the shell to the mass. It therefore follows from this that the gravitational force felt by a mass located inside/outside a uniform spherical shell can be shown to be equal to the portion of the mass of the sphere contained within the radius r. Mathematically this can expressed with

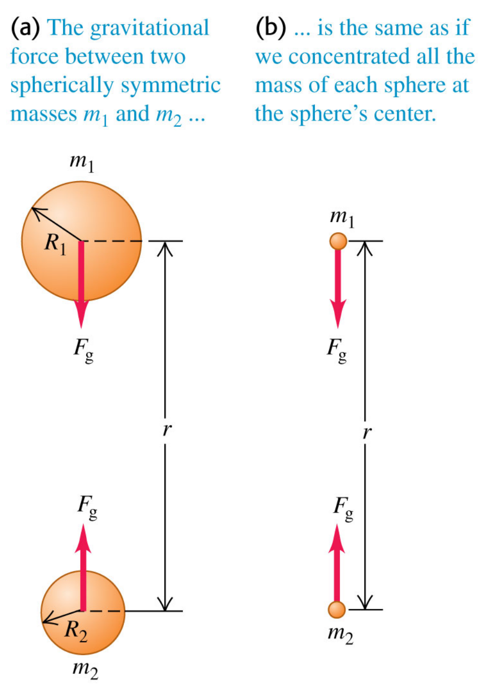

Figure 1 – Gravitational interaction between two spherically symmetric bodies.

$$\mathbf{F}_{\mathrm{grav}} = \begin{cases} -\dfrac{GmMr}{R^3}\mathbf{e}_r, & r < R \\ -\dfrac{GmM}{r^2}\mathbf{e}_r, & r \geq R. \end{cases} \tag{5.5}$$

The result when the mass is located inside the sphere, i.e., when $r < R$, can be explained by considering the relation between the mass M, the mass density ρ (units of $\mathrm{kg/m^3}$), and the volume V (units of $\mathrm{m^3}$) of the sphere. More precisely, we have that

$$M = \rho V, \tag{5.6}$$

with

$$V = \frac{4}{3}\pi R^3 \tag{5.7}$$

for a sphere. If the sphere is uniform, then the density ρ is constant throughout. It follows that the mass M_r contained within a volume V_r (with a radius $r \leq R$) is

$$\begin{aligned} M_r &= \rho V_r \\ &= \rho\left(\frac{4}{3}\pi r^3\right) \\ &= \rho\left(\frac{4}{3}\pi R^3\right)\left(\frac{r}{R}\right)^3 \\ &= \rho V\left(\frac{r}{R}\right)^3, \end{aligned} \tag{5.8}$$

or from equation (5.6)

$$M_r = M\left(\frac{r}{R}\right)^3. \tag{5.9}$$

We can then use equation (5.3) to write down the force felt by the mass when it is located inside the sphere (when $r \leq R$)

$$\begin{aligned} \mathbf{F}_{\mathrm{grav}} &= -\frac{GmM_r}{r^2}\mathbf{e}_r \\ &= -\frac{GmM}{r^2}\left(\frac{r}{R}\right)^3\mathbf{e}_r \end{aligned} \tag{5.10}$$

and

$$\mathbf{F}_{\text{grav}} = -\frac{GmMr}{R^3}\mathbf{e}_r. \tag{5.11}$$

We thus recover the first of equations (5.5).

5.1.2 Exercises

1. (Prob. 13.2 in Young and Freedman.) For the **Cavendish experiment**, the balance apparatus shown in Figure 2 has masses $m_1 = 1.10\ \text{kg}$ and $m_2 = 25.0\ \text{kg}$, while the rod connecting the m_1 pairs is $l = 30.0\ \text{cm}$ long. If, in each pair, m_1 and m_2 are $r = 12.0\ \text{cm}$ apart centre-to-centre, find (a) the net force and (b) the net torque (about the rotation axis) on the rotating part of the apparatus. (c) Does it seem that the torque in part (b) would be enough to easily rotate the rod?

Solution.

(a) For each pair of spheres we have an interaction force

$$F_{\text{grav}} = \frac{Gm_1m_2}{r^2}, \tag{5.12}$$

but directed in opposite directions. The net force is therefore zero.

(b) The net torque is given by

$$\tau = 2F_{\text{grav}}l, \tag{5.13}$$

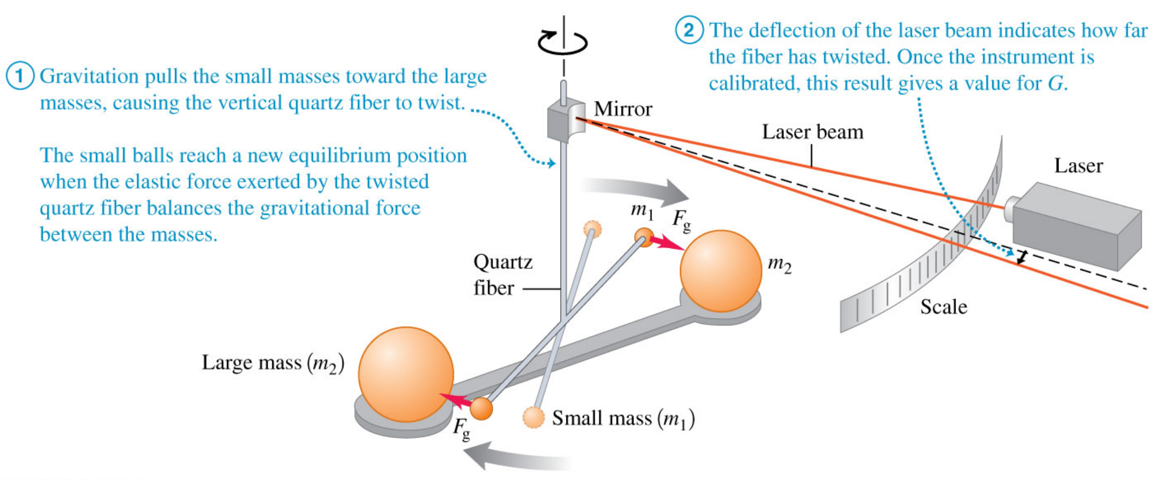

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Figure 2 – The apparatus used by Cavendish to measure the magnitude of the universal gravitational constant.

which yields

$$\begin{aligned}\tau &= 2\frac{Gm_1m_2}{r^2}l \\ &= 2\cdot\frac{6.67\times10^{-11}\mathrm{N\cdot m^2/kg^2}\cdot 1.10\ \mathrm{kg}\cdot 25.0\ \mathrm{kg}}{(0.12\ \mathrm{m})^2}\cdot 0.30\ \mathrm{m} \\ &= 7.64\times10^{-8}\ \mathrm{N\cdot m}.\end{aligned} \tag{5.14}$$

(c) The torque is exceedingly small, which implies that the apparatus (and the quartz fibre) must be exquisitely sensitive. Increasing the mass of the spheres or the length of the lever arm, or decreasing the separation between m_1 and m_2 would increase the torque.

2. (Prob. 13.7 in Young and Freedman.) A typical adult human has a mass of about 70.0 kg. (a) What force does a full moon exert on such a human when it is directly overhead with its centre 380,000 km away? (b) Compare this force with that exerted on the human by the earth.

Solution.

(a) The mass of the moon is $m_\mathrm{M} = 7.35\times10^{22}\ \mathrm{kg}$, and the force exerted by the moon is

$$\begin{aligned}F_\mathrm{moon} &= \frac{Gm_\mathrm{M}m}{R_\mathrm{M}^2} \\ &= \frac{6.67\times10^{-11}\mathrm{N\cdot m^2/kg^2}\cdot 70.0\ \mathrm{kg}\cdot 7.35\times10^{22}\ \mathrm{kg}}{\left(3.78\times10^{8}\mathrm{m}\right)^2} \\ &= 2.40\times10^{-3}\mathrm{N}.\end{aligned} \tag{5.15}$$

(b) We could start by calculating the force exerted by the earth by calculating the weight $w = F_\mathrm{earth} = mg$, but we will instead calculate the ratio of the two forces with

$$\begin{aligned}\frac{F_\mathrm{moon}}{F_\mathrm{earth}} &= \frac{Gm_\mathrm{M}m}{R_\mathrm{M}^2}\bigg/\frac{Gm_\mathrm{E}m}{R_\mathrm{E}^2} \\ &= \left(\frac{R_\mathrm{E}}{R_\mathrm{M}}\right)^2\left(\frac{m_\mathrm{M}}{m_\mathrm{E}}\right) \\ &= \left(\frac{6.37\times10^{6}\mathrm{m}}{3.80\times10^{8}\mathrm{m}}\right)^2\left(\frac{7.35\times10^{22}\mathrm{kg}}{5.97\times10^{24}\mathrm{kg}}\right) \\ &= 3.46\times10^{-6},\end{aligned} \tag{5.16}$$

where $M_\mathrm{E} = 5.97\times10^{24}\mathrm{kg}$ and $R_\mathrm{E} = 6.37\times10^{6}\mathrm{m}$ are the mass and mean radius of the earth, respectively.

5.2 Weight and Gravitational Potential Energy

In previous chapters we modeled the force exerted by the earth on a particle of mass m by its weight

$$w = mg, \tag{5.17}$$

with g the gravitational acceleration due to the earth. Referring to Problem 2 above, we can now easily evaluate this quantity by equating the weight with the gravitational force exerted by the earth at its surface. That is,

$$mg = \frac{Gm_{\mathrm{E}}m}{R_{\mathrm{E}}^2}, \tag{5.18}$$

or

$$\begin{aligned} g &= \frac{Gm_{\mathrm{E}}}{R_{\mathrm{E}}^2} \\ &= \frac{6.67\times10^{-11}\,\mathrm{N}\cdot\mathrm{m}^2/\mathrm{kg}^2 \cdot 5.97\times10^{24}\,\mathrm{kg}}{\left(6.37\times10^{6}\,\mathrm{m}\right)^2} \\ &= 9.81\ \mathrm{m/s}^2, \end{aligned} \tag{5.19}$$

as was expected. It should however be clear, and it is important to note, that *the gravitational acceleration and force exerted by the earth vary with the position of the particle above the surface of the earth; they both decrease as the particle is located further away from the surface (or the centre) of the earth*.

In Section 2.7 of Chapter 2 we also saw that the gravitational force is conservative. That is, the change of potential energy associated with the gravitational force is only a function of the initial and final positions of the path traced by the particle. This in turn means that the gravitational force is (minus) the gradient of the **gravitational potential energy** U_{grav}

$$\mathbf{F}_{\mathrm{grav}} = -\nabla U_{\mathrm{grav}}. \tag{5.20}$$

For a particle located at a position $r \geq R_{\mathrm{E}}$ the gravitational force exerted by the earth is

$$\mathbf{F}_{\mathrm{grav}} = -\frac{Gm_{\mathrm{E}}m}{r^2}\mathbf{e}_r, \tag{5.21}$$

where m is once again the mass of the particle and $\mathbf{e}_r$ is the unit vector pointing from the centre of the earth to the position occupied by the particle. Combining equations (5.20) and (5.21) we write

$$\frac{dU_{\text{grav}}}{dr} = \frac{Gm_{\text{E}}m}{r^2}. \tag{5.22}$$

It can readily be verified that

$$U_{\text{grav}} = -\frac{Gm_{\text{E}}m}{r} \tag{5.23}$$

is a solution consistent with equations (5.20) and (5.22). Equation (5.23) is however different to the relation we have used so far for the gravitational potential energy

$$U_{\text{grav}} = mg\Delta r, \tag{5.24}$$

where the quantity Δr can be conveniently chosen to be the position of the particle *relative to* the surface of the earth. We will now establish that this apparent disparity is solely due to our previously implied assumption that the particle is located very near the surface of the earth, where the gravitational acceleration can, to a good approximation, considered constant at g.

Let us then assume that

$$r = R_{\text{E}} + \Delta r, \tag{5.25}$$

with $\Delta r \ll R_{\text{E}}$. Inserting equation (5.25) in equation (5.23) yields

$$\begin{aligned} U_{\text{grav}} &= -\frac{Gm_{\text{E}}m}{R_{\text{E}} + \Delta r} \\ &= -\frac{Gm_{\text{E}}m}{R_{\text{E}}\left(1 + \Delta r / R_{\text{E}}\right)} \\ &\simeq -\frac{Gm_{\text{E}}m}{R_{\text{E}}}\left(1 - \frac{\Delta r}{R_{\text{E}}}\right), \end{aligned} \tag{5.26}$$

since when $\Delta r \ll R_{\text{E}}$

$$\frac{1}{1 + \Delta r / R_{\text{E}}} \simeq 1 - \frac{\Delta r}{R_{\text{E}}}. \tag{5.27}$$

We now modify equation (5.26) to

$$U_{\text{grav}} \simeq -\frac{Gm_{\text{E}}m}{R_{\text{E}}} + m\left(\frac{Gm_{\text{E}}}{R_{\text{E}}^2}\right)\Delta r. \tag{5.28}$$

We finally make two observations: *i)* the quantity in parentheses in the second term on the right-hand side of equation (5.28) is simply the gravitational acceleration at the surface of the earth (see equation (5.19)) and *ii)* as was stated previously in Section 2.7 of Chapter 2 the potential gravitational energy can only been defined up to constant, and thus the first term on the right-hand side of equation (5.28), which is constant, can be ignored (if one wishes) and the potential gravitational energy near the surface of the earth redefined to

$$U_{\text{grav}} \simeq mg\Delta r. \tag{5.29}$$

This justifies the relation we have used so far.

Referring back to our analysis of Section 2.7 in Chapter 2, we could have easily derived equation (5.23) for the gravitational potential energy by considering the work done by the gravitational force on the particle (of mass m). That is, when the particle moves from a point $\mathbf{r}_1$ to another point at $\mathbf{r}_2$ in the earth's gravitational field we have

$$\begin{aligned} W &= \int_1^2 \mathbf{F}_{\text{grav}} \cdot d\mathbf{r} \\ &= -Gm_{\text{E}}m \int_1^2 \frac{\mathbf{e}_r}{r^2} \cdot d\mathbf{r} \\ &= -Gm_{\text{E}}m \int_1^2 \frac{dr}{r^2} \\ &= \frac{Gm_{\text{E}}m}{r_2} - \frac{Gm_{\text{E}}m}{r_1}, \end{aligned} \tag{5.30}$$

since $\int dr/r^2 = -1/r$. Using equation (5.23) we then write

$$W = U_{\text{grav}}(\mathbf{r}_1) - U_{\text{grav}}(\mathbf{r}_2), \tag{5.31}$$

which is perfectly consistent with the corresponding equation (i.e., equation (2.99)) we derived in Section 2.7. This once again verifies that the gravitational force is indeed conservative, since the change in potential energy is not a function of the path taken between the initial and final points. Although these results were derived using the gravitational interaction with the earth as an example, they apply to any body in general.[1]

5.2.1 Exercises

3. **The Escape Speed.** Find the minimum initial velocity needed to eject a projectile of mass m up and away from the gravitational attraction of the earth.

[1] We must make an exception for bodies that are extremely dense (e.g., black holes), for which Newton's theory of gravitation fails and Einstein's Theory of General Relativity must be used.

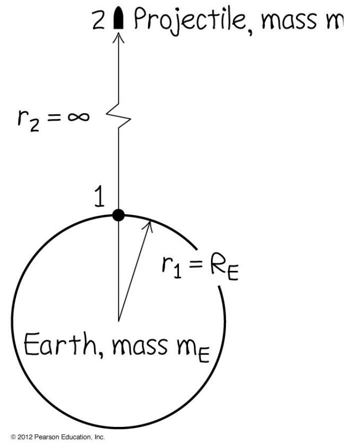

Figure 3 – Escape of a projectile away from the earth's gravitational pull.

Solution.

Because the gravitational force is conservative, the mechanical energy of the system must be conserved. If we denote by "1" and "2" the conditions at the moment the mass is ejected and when it is away from the gravitational attraction of the earth, respectively, we then write

$$\begin{aligned} K_1 + U_{\text{grav},1} &= K_2 + U_{\text{grav},2} \\ \frac{1}{2}mv_1^2 - \frac{Gm_\text{E}m}{r_1} &= \frac{1}{2}mv_2^2 - \frac{Gm_\text{E}m}{r_2}. \end{aligned} \tag{5.32}$$

But since we are looking for the "minimum" speed, we set $v_2 = 0$. Furthermore, to be completely exempt from the gravitational attraction of the earth implies that $r_2 = \infty$. We then find from the second of equations (5.32) that

$$\begin{aligned} v_1 &= \sqrt{\frac{2Gm_\text{E}}{r_1}} \\ &= 1.12 \times 10^4 \text{ m/s } (= 40{,}200 \text{ km/h}) \end{aligned} \tag{5.33}$$

in the vertical direction and where $r_1 = R_\text{E}$ (see Figure 3).

4. (Prob. 3.15 in Young and Freedman.) Calculate the earth's gravity force on a 75.0 kg astronaut who is repairing the Hubble Space Telescope 600 km above the earth's surface, and then compare this value with his weight at the earth's surface. In view of your result, explain why we say that astronauts are weightless when they orbit the earth in a satellite

such as the space shuttle. Is it because the gravitational pull of the earth is negligibly small?

Solution.

The location of the astronaut relative to the centre of the earth is $r_1 = R_E + 6.00 \times 10^5$ m. Using equation (5.21) we have

$$\begin{aligned} F_{\text{grav}} &= \frac{Gm_E m}{r_1^2} \\ &= \frac{6.67 \times 10^{-11}\,\text{N} \cdot \text{m}^2/\text{s}^2 \cdot 5.97 \times 10^{24}\,\text{kg} \cdot 75.0\text{ kg}}{\left(6.97 \times 10^6\,\text{m}\right)^2} \\ &= 610\text{ N.} \end{aligned} \tag{5.34}$$

While on the earth, according to equation (5.17)

$$\begin{aligned} w &= mg \\ &= 75.0\text{ kg} \cdot 9.81\text{ m/s}^2 \\ &= 736\text{ N.} \end{aligned} \tag{5.35}$$

The gravitational pull on the astronaut is therefore not negligible. However, both the Hubble Space Telescope and the astronaut feel the same acceleration Gm_E/r^2 and therefore don't accelerate relative to one another, but they also are also on the same orbit above the earth and are thus constantly "free falling" and do not feel their weight.

5. **Circular Orbits.** A satellite is launched horizontally (relative to the surface of the earth) from a distance r (i.e., a radius) from the centre of the earth. Determine the launch speed v necessary for the satellite to enter into a circular orbit about the earth. Calculate the mechanical energy of the orbiting satellite assuming that the gravitational potential energy is zero at $r = \infty$. Assume that the orbit is located above the earth's atmosphere such that there is no air friction on the satellite.

Solution.

We know from our work in Section 1.4 of Chapter 1 on circular motions that the centripetal acceleration for a particle moving at a speed v at a radius r is given by (see Figure 4)

$$a_{\text{rad}} = \frac{v^2}{r}. \tag{5.36}$$

Applying Newton's Second Law we have

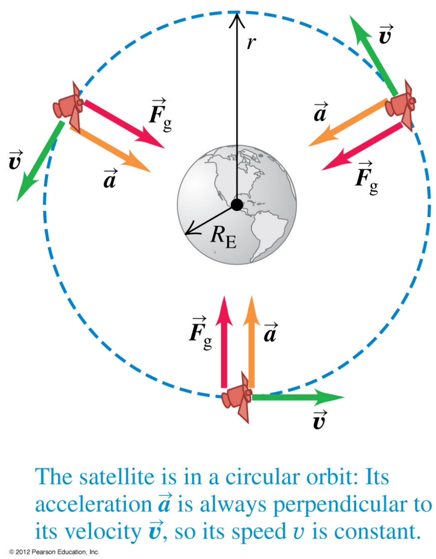

Figure 4 – Satellite on a circular orbit around the earth.

$$ma_{\text{rad}} = \frac{Gm_E m}{r^2}, \tag{5.37}$$

or

$$v = \sqrt{\frac{Gm_E}{r}}. \tag{5.38}$$

The energy of the orbiting satellite is

$$\begin{aligned} E &= K + U \\ &= \frac{1}{2}mv^2 - \frac{Gm_E m}{r}, \end{aligned} \tag{5.39}$$

where it is clear that the potential energy is zero at infinity (remember that the potential energy is defined only "up to a constant"). Inserting equation (5.38) into equation (5.39) yields

$$\begin{aligned} E &= \frac{Gm_E m}{2r} - \frac{Gm_E m}{r} \\ &= -\frac{Gm_E m}{2r} \\ &< 0. \end{aligned} \tag{5.40}$$

It is interesting to note that the energy of the satellite is negative. This is true of any particle on a bound orbit, when the potential energy is defined as being zero at infinity (e.g., an electron orbiting the nucleus in an atom). We can verify that the ejected projectile of Problem 3, which has a speed $v = 0$ at infinity, has zero total mechanical energy.

5.3 Kepler's Laws

Several decades before Newton enunciated his Law of Gravitation, Johannes Kepler used precise astronomical data of planetary motions to empirically deduce three general laws governing the orbits of the planets orbiting the sun. These three laws are as follows:

I. *Planets move on elliptical orbits about the sun with the sun at one focus.*
II. *The area per unit time swept out by a radius vector from the Sun to a planet is constant.*
III. *The square of a planet's orbital period is proportional to the cube of the major axis of the planet's orbit.*

These laws were all, in time, rigorously explained by Newton. But the derivation of the proof of Kepler's First Law, in particular, requires a level of mathematical sophistication that is beyond the scope of our studies. An example of an elliptical planetary orbit is shown in Figure 5, where the sun is located at the focus S. The orbit is characterized by its **semi-major axis** a, which is a measure of the size of the orbit, and its **eccentricity** e. The distance of the two foci from the centre of the ellipse is given by $\pm ea$ on the major axis line. It is interesting to note that the orbits of planets in the solar system have relatively modest eccentricities, ranging from 0.007 for Venus and 0.206 for Mercury. The earth has an eccentricity of 0.017, while an eccentricity of zero corresponds to a circular orbit. The closest and furthest orbital points to the sun cross the major axis at the **perihelion** and the **aphelion**, respectively.

For Kepler's Second Law we can start by considering the torque acting on the planet

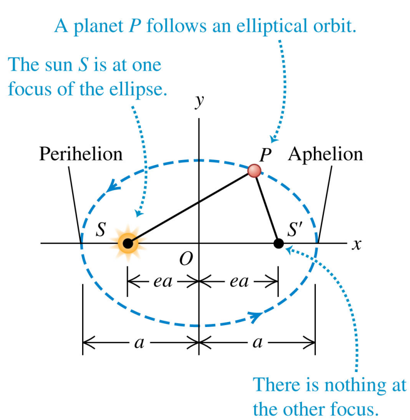

Figure 5 – The geometry of an elliptical planetary orbit about the sun. The sun is located at one of the foci of the ellipse.

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}_{\text{grav}}, \tag{5.41}$$

where also, as usual, $\boldsymbol{\tau} = d\mathbf{L}/dt$.

However, since the vector $\mathbf{r}$ defining the position of the planet in relation to the sun and the gravitational pull of the sun on the planet $\mathbf{F}_{\text{grav}}$ are collinear, the torque on the planet is zero. It follows that the angular momentum of the planet is conserved.

Considering the definition for the angular momentum

$$\begin{aligned} \mathbf{L} &= \mathbf{r} \times \mathbf{p} \\ &= mrv\sin(\phi)\mathbf{e}_{\perp} \\ &= \text{constant}, \end{aligned} \tag{5.42}$$

where $\mathbf{e}_{\perp}$ is the unit vector perpendicular to orbital plane, m is the planet's mass, and ϕ is the angle between planet's linear momentum vector $\mathbf{p}$ and $\mathbf{r}$ (see Figure 6). Since the orbital motion takes place in a plane, we can use

$$\begin{aligned} v_{\perp} &\equiv v\sin(\phi) \\ &= \omega r, \end{aligned} \tag{5.43}$$

to transform equation (5.42) to

$$\begin{aligned} \frac{L}{m} &= r^2\omega \\ &= r^2\frac{d\theta}{dt}, \end{aligned} \tag{5.44}$$

since $\omega = d\theta/dt$. However, the triangular area dA swept as the planet travels an infinitesimal angle $d\theta$ is one-half the product of the radius r and the arc $rd\theta$

$$dA = \frac{1}{2}r^2 d\theta. \tag{5.45}$$

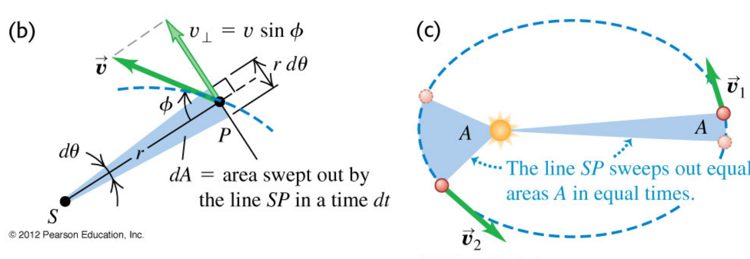

Figure 6 – The area swept by a planet orbiting the sun in a given time interval.

Equation (5.44) then becomes

$$\frac{L}{2m} = \frac{dA}{dt}, \tag{5.46}$$

but since the angular momentum is conserved *the area swept per unit time* is found to agree with Kepler's Second Law, i.e.,

$$\frac{dA}{dt} = \text{constant}. \tag{5.47}$$

It also follows that the total area A swept over a finite interval $T_2 - T_1$ is constant since

$$\begin{aligned} A &= \int_{T_1}^{T_2} dA \\ &= \frac{L}{2m}\int_{T_1}^{T_2} dt \\ &= \frac{L}{2m}\left(T_2 - T_1\right). \end{aligned} \tag{5.48}$$

It therefore does not matter what part of the orbit the planet travels during the interval, the total area swept is the same (see Figure 6).

Equation (5.48) can also be used to demonstrate Kepler's Third Law. While the general derivation for an elliptical orbit is somewhat complicated, but we can easily verify it for circular orbits. To do so we consider the total area of the circle as the planet completes one full orbit about the sun. If we define the orbital period as T, while realizing that the area contained within a circle of radius (or semi-major axis) a is $A = \pi a^2$, then

$$T = \pi a^2 \frac{2m}{L}. \tag{5.49}$$

For a circular orbit $\phi = \pi/2$ and equation (5.42) yields (with $r = a$)

$$\begin{aligned} \frac{L}{m} &= av \\ &= \sqrt{Gm_{\mathrm{E}}a}, \end{aligned} \tag{5.50}$$

where we used equation (5.38) for the orbital speed. Equation (5.49) then becomes

$$T^2 = \frac{4\pi^2 a^3}{Gm_{\mathrm{E}}}, \tag{5.51}$$

and Kepler's Third Law is verified for circular orbits.

5.4 Exercises

6. (Prob. 13.28 in Young and Freedman.) In 2004 astronomers reported the discovery of a large Jupiter-sized planet orbiting very close to the star HD 179949 (hence termed a "hot Jupiter"). The orbit was just $1/9$ the distance of Mercury from our sun, and it takes the planet only 3.09 days to make one orbit (assumed to be circular). (a) What is the mass of star? (b) How fast is this planet moving?

Solution.

(a) According to equation (5.51) the mass of the star is given by

$$\begin{aligned} m_* &= \frac{4\pi^2 a^3}{GT^2} \\ &= \frac{4\pi^2 \cdot \left(1/9 \cdot 5.79\times10^{10}\,\mathrm{m}\right)^3}{6.67\times10^{-11}\,\mathrm{N\cdot m^2/kg^2} \cdot \left(3.09\cdot 86{,}400\ \mathrm{s}\right)^2} \\ &= 2.21\times10^{30}\ \mathrm{kg}\ \left(1.11\ \mathrm{M}_\odot\right), \end{aligned} \tag{5.52}$$

where the mean orbital radius of Mercury is $5.79\times10^{10}\,\mathrm{m}$, one *solar day* has 86,400 s, and the mass of the sun is $1\ \mathrm{M}_\odot = 1.99\times10^{30}\ \mathrm{kg}$.

(b) The orbital speed of the planet is

$$\begin{aligned} v &= \frac{2\pi r}{T} \\ &= \frac{2\pi \cdot \left(1/9 \cdot 5.79\times10^{10}\,\mathrm{m}\right)}{3.09\cdot 86{,}400\ \mathrm{s}} \\ &= 151\ \mathrm{km/s}. \end{aligned} \tag{5.53}$$

7. **True and Apparent Weights.** Because of the earth's rotation the apparent weight w of an object depends on the latitude λ at which it is measured (e.g., with a scale). In contrast, the true weight is entirely due to the earth's gravity and its magnitude equals $w_0 = mg_0$ and is the same everywhere (m is the mass of the object). (a) If the tangential speed of a point on the surface of the earth at the equator is v, then find the apparent weight of the object at the equator. (b) What is the apparent gravitational acceleration at the equator?

Solution.

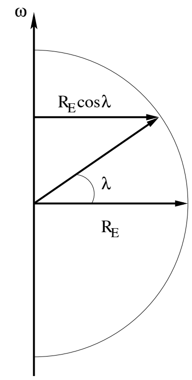

Figure 7 - A section of the earth for Prob. 7.

(a) Let us denote the angular speed of the earth with ω. Then the speed of a point on the surface of the earth at latitude λ is (see Figure 7).

$$v(\lambda) = \omega R_{\mathrm{E}} \cos(\lambda). \tag{5.54}$$

Applying Newton's Second Law for an object located at the equator on the surface of the earth we have

$$\begin{aligned} mg_0 - F &= ma_{\mathrm{rad}} \\ &= m\frac{v^2}{R_{\mathrm{E}}} \\ &= m\omega^2 R_{\mathrm{E}}, \end{aligned} \tag{5.55}$$

where F is the "normal" force the earth exerts on the object. The apparent weight is

$$w = m\left(g_0 - \omega^2 R_{\mathrm{E}}\right). \tag{5.56}$$

(b) The apparent gravitational acceleration at the equator is

$$\begin{aligned} g &= g_0 - \omega^2 R_{\mathrm{E}} \\ &= 9.81 \text{ m/sec}^2 - \left(\frac{2\pi}{86{,}164 \text{ s}}\right)^2 \cdot 6.37 \times 10^6 \text{ m} \\ &= \left(9.81 - 3.39 \times 10^{-2}\right) \text{m/sec}^2 \\ &= 9.78 \text{ m/sec}^2. \end{aligned} \tag{5.57}$$

Please note that although there are 86,400 s in one *solar day*. The earth completes one rotation in $(1 - 1/365)$ solar day, since it is traveling 1/365th of its orbit during that time (i.e., the rotation period of the earth is 86,164 s long).

8. **Black Holes.** Given a very massive star of mass M, calculate the radius (called the **Schwarzschild Radius**) for which not even light emanating from the surface of the star can escape. Determine the Schwarzschild radius of the sun and the mean mass density it would have if its mass were contained in the corresponding spherical volume (the spherical surface bounding this volume is called the **event horizon**).

Solution.

To calculate the Schwarzschild radius we use equation (5.33) for our star and replace the escape speed v by c, the speed of light,

$$c = \sqrt{\frac{2GM}{R_S}} \tag{5.58}$$

or

$$R_S = \frac{2GM}{c^2}. \tag{5.59}$$

For the sun, we have

$$\begin{aligned} R_S &= \frac{2 \cdot 6.67 \times 10^{-11} \text{N} \cdot \text{m}^2/\text{s}^2 \cdot 1.99 \times 10^{30} \text{kg}}{\left(3.00 \times 10^8 \text{m/s}\right)^2} \\ &= 2{,}950 \text{ m}. \end{aligned} \tag{5.60}$$

The Schwarzschild radius is evidently much smaller than the radius of the sun (i.e., $R_\odot = 6.96 \times 10^8$ m). If the sun were all contained within the event horizon, then its mean mass density would be

$$\begin{aligned} \rho &= \frac{M}{4\pi R_S^3/3} \\ &= \frac{3 \cdot 1.99 \times 10^{30} \text{kg}}{4\pi \cdot \left(2{,}950 \text{ m}\right)^3} \\ &= 1.84 \times 10^{19} \text{kg/m}^3, \end{aligned} \tag{5.61}$$

which is $R_\odot^3 / R_S^3 = 1.31 \times 10^{16}$ times greater than the actual mean mass density of the sun. We note that even though equation (5.59) is correct, it should not be derived using Newton's Law of Gravitation, as we did, instead Einstein's Theory of General Relativity must be used. The fact the right result was obtained is due to a fortuitous cancelling of errors…